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Quaternionic Projective Theory and Hadron Transformation Laws

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A quaternionic projective theory based on the symmetry group Sl(2,**H**) allows one to identify various hadron models and many well-known particle transformation laws in its subgroup chains. Identifying the 16-dimensional Dirac algebra $\{\gamma^\mu\}$ with Sl(2,**H**), we use a well-established group-theoretic framework as well as the framework of projective geometry to classify elementary particles and describe their interactions at low energies. It is straightforward to derive Chiral Dynamics and explain the spinorial ('quark') structure of hadrons. Spontaneous symmetry breaking occurs naturally by coset reductions, whereas 'classical' physics is obtained via well-defined limits in terms of a group contraction. The Dirac equation can be identified within a Riemannian globally symmetric space and thus allows one to investigate the fermionic mass as a well-defined parameter. In addition, we suggest an identification of the second quantization scheme and an approach to sum up the perturbation series.

1. HADRON MODELS AND PHENOMENOLOGY

In hadron physics, numerous phenomenological models are used to describe various aspects of hadronic interactions. Almost all relativistic approaches to hadron physics classify the nucleon as a fundamental Dirac spinor ψ with appropriate (complex) components and introduce its (relativistic) interactions as coefficients $\{s, p, v_\mu, a_\mu, f_{\mu\nu}\}$ of an element Γ within the Dirac algebra $\{\gamma^\mu\}$, i.e.,

$$\Gamma = s1_{4\times 4} + p\gamma_5 + v_\mu\gamma^\mu + a_\mu\gamma_5\gamma^\mu + f_{\mu\nu}\sigma^{\mu\nu} \qquad (1)$$

The coefficients themselves are determined by physical assumptions of the respective models. Electromagnetic interactions of fermions are described by

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minimal substitution/gauge coupling of the photon field A^{μ} to the charge $\hat{q}$ of a Dirac spinor; we identify $v_{\mu} = -\hat{q}\, A_{\mu}$ in Eq. (1). The appropriate ('gauge') Langrangian for the fundamental representation ψ then reads

$$\mathcal{L}^{g} = -\overline{\psi}\hat{q}\, A_{\mu}\gamma^{\mu}\psi \tag{2}$$

Although this Lagrangian seems to be well suited to describe the behavior of the electron, it does not describe the nucleon. If we associate the nucleon with the fundamental Dirac spinor ψ, already in lowest order perturbation theory it is necessary to introduce large corrections to the gauge coupling scheme by adding the 'Pauli term' (Bjorken and Drell, 1966). These phenomenological corrections parametrize electromagnetic nucleon properties by additional tensorial coefficients $F^{\mu\nu}$,

$$\mathcal{L}^{em}_{\gamma NN} = \mathcal{L}^{g} - e\overline{\psi}\,\frac{\hat{\kappa}}{4m}\,\sigma_{\mu\nu}F^{\mu\nu}\psi \tag{3}$$

where $\hat{\kappa}$ is the isospin operator related to the anomalous magnetic moment of the nucleon.

The numerous approaches towards a field theory of strong interactions are based on further identifications of coefficients in Eq. (1), the first and simplest using only $p = -ig_0\,\vec{\tau}\cdot\vec{\pi}$ in Eq. (1). This coefficient p not only parametrizes a pseudoscalar quantity, but it also comprises a su(2) vectorial element $\vec{\tau}$ which phenomenologically serves to introduce the pion fields $\vec{\pi}$ and additional (twofold) spinorial isospin structure into the spinorial representation ψ of the nucleon. However, the related pseudoscalar Lagrangian (Bjorken and Drell, 1966)

$$\mathcal{L}_{\pi NN} = -ig_0\overline{\psi}\gamma_5\vec{\tau}\cdot\vec{\pi}\psi \tag{4}$$

is not suitable to describe the observable pion–nucleon interaction because it yields too strong s-wave pions at low energies. Moreover, the isomultiplets in the energy spectrum show mass splitting as well as strong transition amplitudes between various SU(2) irreps, e.g., the nucleon and the delta resonances, so that this observation does not fit to a Wigner–Weyl realized (compact) symmetry group SU(2). Thus, a simple (pseudoscalar) SU(2) symmetry scheme is suitable neither for static multiplet classification nor for a description of dynamic (interacting) states.

Within a more sophisticated ansatz, the nucleon and pion degrees of freedom have been associated with SU(2) multiplets emerging from a spontaneous breakdown of a 'chiral' SU(2) $\times$ SU(2) symmetry group to its diagonal subgroup $SU_V(2)$. A special (nonlinear) realization of such a Lagrangian is given by (Weinberg, 1968)

$$\mathscr{L}_{\pi NN} = \frac{f_{\pi NN}}{m_\pi} \bar{\psi}\gamma_\mu\gamma_5\vec{\tau} \cdot \partial^\mu\vec{\pi}\psi - \lambda^2\bar{\psi}\gamma_\mu\vec{\tau} \cdot (\vec{\pi} \wedge \partial^\mu\vec{\pi})\psi \tag{5}$$

thus occupying the coefficients a_μ and v_μ, respectively, in Eq. (1). Although this (effective) Lagrangian *is constructed* to produce correct pion scattering lengths, it is valid only at very low energies and has similar problems with the particle spectrum, with transition amplitudes, and with parity degeneracy of multiplets as the pseudoscalar Lagrangian (4). In addition, the Goldstone character of the pion fields as required within 'chiral' approaches is in clear contradiction with *massive* pions observed in the spectrum as well as with some theorems known from axiomatic quantum field theory (see, e.g., Fabri and Picasso, 1966; Fabri *et al.*, 1967; (Joos and Weimar, 1976), so that already the underlying symmetry structure of such models, which is based on an exact SU(2) $\times$ SU(2) chiral symmetry group and its spontaneous and additional explicit symmetry breaking, is questionable. Moreover, nonlinear field theories with this physical interpretation are not renormalizable, so that the interpretation of higher loop contributions (see, e.g., the extensive literature on chiral perturbation theory) has no strict mathematical foundation from quantum field theory. Using the Lagrangian (5) as a representative, the nonrenormalizability of these chiral approaches is obvious from a direct calculation of the chiral matrix elements $\mathcal{M}^{\text{ch}}$ in second-order perturbation theory (see Dahm, 1996, for details) in that

$$\mathcal{M}^{\text{ch}} = \mathcal{M}^{\text{iso}} + (\text{nonpole}) \tag{6}$$

The physically relevant pole structure of the invariant chiral matrix elements in the energy plane is solely given by the pole structure $\mathcal{M}^{\text{iso}}$ of the pseudoscalar pion theory, i.e., only by nucleon propagators in *s*- and *u*-channels, whereas the (chiral) derivative terms of the effective pion *p*-wave coupling introduce additional (nonpole) contributions which are of power 0 or higher in the energy. Although these effective energy contributions in the chiral model correct the pion scattering lengths phenomenologically for the matrix elements of second-order perturbation theory, they destroy the integrability of the perturbation series in the energy plane. Thus, all the mostly phenomenological constructions of chiral theories suffer heavily already from the underlying symmetry concept and its realization on complex spinor spaces.

From both approaches, however, we want to adopt the following requirements for our further treatment of hadrons: Although we obviously find strong transitions between irreducible SU(2) isospin representations in the particle spectrum so that the concept of an exactly realized compact SU(2) flavor symmetry does not hold, the symmetry group we use later as a basis for classification and interactions has to have particle representations such that SU(2) isospin multiplets *seem* to be realized. Furthermore, when identifying the (p-wave) pion as meson within the regular representation of a (compact) Lie group, the nucleon and delta both have to be members of the same irreducible fermionic group representation to justify the new symmetry approach. In Section 4, we show that both requirements are satisfied by the groups SU(4) resp. SU*(4).

A third phenomenological way of describing hadrons comprises the many approaches via 'quark models' or by QCD as a generalization of the gauge principle to higher compact symmetry groups. Common to all these efforts is the idea to introduce a substructure into the theoretical Dirac description of a SU(2) flavor nucleon by hand in order to explain a (then necessary) 'spatial extension', other phenomenological corrections to the Dirac description, deep inelastic experiments, etc. We shall show in Section 4 that the spinorial quark structure when interpreted as 'substructure' of hadron representations emerges automatically within the spinor representations of SU(4) resp. SU*(4) on complex representation spaces. Moreover, by identifying the matrices $\{\gamma^\mu\}$ within the coset decomposition $Sl(2,\mathbf{H})/Sp(2) \cong SU^*(4)/USp(4)$, it becomes apparent that the role of the vectorial information v_μ in the Lagrangian is determined by geometry. Like in common gauge theories, the gauge 'field' A^μ can be added (infinitesimally) to the momentum p^μ to explain δp^μ in a dynamic picture; however, within our approach there is no further necessity to introduce noncommutative group structures and additional 'fields' by means of a gauge group besides SU*(4) spinorial structures to describe (hadronic) massive matter fields. The odd-rank spinor representations describe matter fields, even-rank spinors describe strong interacting massive bosons, and 'classical relativity' emerges after group contraction (Dahm, 1997a).

At this point, we have to discuss an additional common problem of quantum field theory in that one has to distinguish very carefully between 'local' infinitesimal fields like gauge fields and integrated ('physical') fields. Although the framework of Lie algebras, Lie groups, and coset spaces is extremely suitable and allows one to distinguish the respective interpretations, usually infinitesimal (gauge) fields and physically observable fields are identified on the same footing within Eq. (1) and thus mixed up. A typical example is the identification of the photon as a gauge boson in v_μ and the additional identification of $\vec{E}$ and $\vec{B}$ in $f_{\mu\nu}$ at the same time. As we later show, the coset decompostion SU*(4)/USp(4) allows us to calculate the general exponential $\exp(V)$ of its (infinitesimal) vectorial elements V [see also Eqs. (7) and (9)] very easily.

Right from the very beginning, we do not want to follow the historical approaches to hadron symmetries as cited above, i.e., by using an underlying relativistic theory with additional symmetry degrees of freedom described in terms of certain compact Lie groups. In all cases, such an approach to hadronic physics makes it necessary to introduce more and more (mainly phenomenological) corrections into an originally rigorous theory, and more and more additional mechanisms and pictures to handle and interpret these corrections. Moreover, we do not want to apply various symmetry-breaking mechanisms to a given compact Lie group because this is usually the manner to introduce additional mathematical difficulties and physical problems into a theory. In summarizing the history mentioned above, the only axiom we have learned up to now from all approaches is the fact that although the electron transforms according to the Dirac spinor in the fundamental representation ψ, *the nucleon does not!* Thus, in the following we present an alternate ansatz towards relativistic theories which describes the nucleon as a higher rank spinorial representation (rep) *other* than ψ. This step allows us to derive hadron properties from pure geometry and in addition allows us to relate to classical physics by means of a well-defined mathematical framework. Starting only from the Dirac algebra and applying Lie theory, we obtain a very suitable description of hadron properties and transformation laws as well as symmetry structures and a classification scheme which exactly yields the spinorial quark structure of the known massive hadron representations in the low-energy regime of the spectrum.

2. QUANTUM FIELD THEORY, GEOMETRY, AND SOME ASSOCIATED LIE GROUPS

In QFT, the fundamental rep ψ describes a relativistic spinor field on which we act with an element of the Dirac algebra $\{\gamma^\mu\}$ to describe its relativistic transformations. This algebra is well known to be a Clifford algebra, as can be seen directly by calculating the anticommutator of the vectorial elements γ^μ. Here, however, we also take into account the *commutator* of the algebra elements $\{\gamma^\mu\}$ to extend the scope of our investigations. It is straightforward to calculate the commutators and see that they close into the Lie algebra sl(2,**H**). Thus, in addition to the benefits of the common Clifford-algebraic approach of QFT, we may use Lie theory to classify particle multiplets and their transformation properties as well as to integrate the perturbation series. Moreover, the related Lie group Sl(2,**H**) can be understood in terms of transformations on twofold homogeneous quaternionic coordinates and thus yields an underlying (global) projective geometry based on the division algebra of quaternions (Dahm, 1995). By embedding the related twofold homogeneous quaternion coordinates into complex number spaces, we obtain the isomorphic noncompact matrix group SU*(4) and the spinors known from Dirac theory (Dahm, 1996).

Besides these nice, but very profound structural features, there are also some immediate and very practical advantages of this approach:

1. In addition to the fundamental rep ψ, the set of possible physical states and their respective transformations is well-defined in terms of representation theory of the group Sl(2,**H**) resp. SU*(4).

2. Γ as given in Eq. (1) transforms according to the regular representation of the groups Sl(2,**H**) resp. SU*(4), and it is straightforward to calculate its action on appropriate spinor representations.

3. Interactions/vertices as defined by Yukawa couplings as well as invariants are defined within the well-known framework of Lie theory resp. projective geometry.

4. The twofold quaternionic projective theory with its 15 ‘physical’ parameters $\{s, p, v_\mu, a_\mu, f_{\mu\nu}\}$ comprises automatically all known quantum field theories, but, due to its global geometric character, is not restricted only to the gauge principle.

5. The complete geometric SU*(4) theory ‘lives’ in curved space-time and thus is not in contradiction with well-known no-go theorems (see, e.g., Dyson, 1966, and references therein). The Poincaré group and the notion of translations emerge only after a group contraction (Dahm, 1997a), i.e., in a special geometrical limit. It is important that in order to discuss this limit in detail, we have to switch from the homogeneous coordinates we discuss subsequently on complex spaces to real affine coordinates and appropriate translations. There are several mathematical mechanisms to achieve this; however, these details are beyond the scope of the subjects covered here.

6. The apparatus of Lie theory allows us to relate infinitesimal algebraic and global (integrated) properties within a well-defined and rigorous framework. Thus, it is ‘only’ a matter of a correct representation theory to identify the physical parameters of relativistic transformations within the calculations and isolate the relevant variables to understand physical laws.

A more exhaustive overview of the geometrical background of our approach is given in Dahm (1995, 1996). For brevity we use the projection S^4 onto $\mathbf{R}^4$ and close by (noncommutative) complexification to **H**. Because **H** is a division algebra, there is *one* point which maps in a well-defined way to infinity. This property ensures that in $\mathbf{R}^4$, any limit to infinity is unique, which is of great importance for physical interpretations. For example, it induces asymptotic boundary conditions in $\mathbf{R}^4$ which introduce naturally a quantization scheme and a scale. Moreover, the introduction of homogeneous quaternionic coordinates allows us to define spinors and leads to the Lie algebra isomorphism sl(2,**H**) $\cong$ su*(4) $\cong$ so(5,1), so that local properties of one and the same (relativistic) physical theory can be handled on three different representation spaces, namely $\mathbf{H}^2$, $\mathbf{C}^4$, and $\mathbf{R}^6$ with appropriate metrics, respectively, by discussing the various real forms of appropriate Lie algebras. Last and not least, this property ensures that observations of microscopical events (like scattering or creation processes) have the same *unique* limit to infinity (i.e., into the measurement apparatus of the observer).

3. CONSEQUENCES

Whereas the quaternionic projective theory is directly related to the Dirac algebra, to biquaternions, and velocity spaces (Dahm, 1995), its complex embedding yields hadronic properties as partially described by spin-flavor supermultiplets or Chiral Dynamics (Dahm, 1995). The representation of quaternionic projective theory on real spaces allows us in a naive but very direct manner to contract the generators of so(5,1) to the generators of the Poincaré group via the de Sitter subalgebra (Dahm, 1995). In addition, however, there exists another possible identification scheme within the coset decomposition $Sl(2,\mathbf{H})/Sp(2) \cong SU^*(4)/USp(4)$. This space offers the foundation for a very deep and profound treatment in that it can be identified as an irreducible Riemannian globally symmetric space of rank 1 and dimension 5. Calculating the basis vectors, one can identify these five elements exactly with the representation of the Dirac equation as given in Dirac (1928). Thus, one obtains a well-defined framework to understand the Dirac equation and especially the identification of the fermionic mass either on the basis of the coset decomposition with respect to the symplectic group or in terms of a hyperbolic (velocity) space which leads to Euclidean geometry by complexifying the mass, i.e., $m^2 \to -m^2$. To investigate the coset decomposition in more detail, we can choose a twofold quaternionic (tensor product) basis $Q_{\alpha\beta} = q_\alpha \times q_\beta$, $Q_{00} \cong \mathbf{1}$, resp. its appropriate representation on $\mathbf{C}_{4\times 4}$, and identify the five basis elements of $SU^*(4)/USp(4)$ as given by $\{iQ_{30}, Q_{2j}, iQ_{10}\}$ (Dahm, 1995). If we then define the operator V by

$$V := \alpha_0 iQ_{30} - i\alpha_j Q_{2j} - aiQ_{10} \tag{7}$$

we immediately obtain a special representation of the Dirac equation (Dirac, 1928) if we set $V\psi = 0$ (which may also be interpreted as choosing a certain equivalence class). Note the artificial complexification of the coset parameters α_j in order to compare to Dirac's ansatz and the metric used there. On the other hand, it is easy to see that the square of the operator V maps to a multiple of unity,

$$V^2 = \{\alpha_0^2 - \vec{\alpha}^2 + a^2\}Q_{00} \tag{8}$$

although the five basis elements $\{iQ_{30}, Q_{2j}, iQ_{10}\}$ of this space in general do *not* commute. A Wick rotation in the zero component α_0 (and of the parameters α_j) suggests a relation of a with an imaginary mass; note, however, that although we are talking about cosets, we are still working with homogeneous coordinates. Nevertheless, by 'misidentifying' the basis vectors underlying Eq. (8) to be Cartesian basis vectors e_j, we immediately have to discuss SO(3,2) as the noncompact symmetry group of this bilinear form; by reidentifying a Wick-rotated 'mass' $m = ia$ the relevant dynamical symmetry group seems to be SO(4,1), or, by complexifying $\alpha_j \rightarrow i\alpha_j$, we obtain SO(5) and may discuss Bhabha equations.

Using Eq. (8), it is now easy to calculate the compete power series of the exponential $\exp(V)$ analytically,

$$\exp(V) = Q_{00} \cosh \alpha + V \frac{\sinh \alpha}{\alpha}, \qquad \alpha^2 = (\alpha_0^2 - \vec{\alpha}^2 + a^2) \tag{9}$$

Thus, it becomes apparent that if we identify the gauge boson A^μ within the parameters of V, the algebraic structrure of the infinitesimal theory $\sim\{V\}$ obviously does not differ algebraically much from the integrated theory $\sim\{Q_{00}, V\}$. On the contrary, it is the exponential mapping that produces exactly terms $\sim Q_{00} = 1$ which do not exist in the Lie algebra, but which we usually introduce by hand to describe masses and charge operators effectively. Note that these properties are in perfect agreement with Lie theory as well as with Clifford theory; however, with respect to a physical interpretation of the respective parameters, we suggest to relate physical observations strictly to the integrated theory. Then, it is just a matter of identifying the coefficients appropriately after integrating over the maximal compact (symplectic) subgroup. For example, after integrating the parameters α and relating them to momenta, the mass parameter has a fixed phase with respect to the momentum and cannot be chosen arbitrarily. Each further complexification/Wick rotation switches to another real form of the Lie algebra, respectively to another Lie group. It is interesting to compare representations in various textbooks and relate them as well as their various calculational results to the respective real forms of the Lie algebra su(4) (to be published).

To summarize our calculations, we suggest to understand a complete relativistic particle theory as described by homogeneous quanternionic coordinates, respectively, by SU*(4), when represented on complex spaces. The requirement of second quantization is equivalent to the subsequent factorization of general SU*(4) transformations with respect to compact USp(4) transformations, i.e., the requirement that the action of an element V of the coset SU*(4)/USp(4) as given in Eq. (7) vanishes when acting on a secondly quantized spinor representation ψ, $V\psi = 0$, yields the Dirac equation. These symplectic transformations preserve an appropriately defined measure. Stated in terms of group theory, the element V is part of the vectorial representation **15** of SU*(4). Thus if ψ represents the fundamental irrep **4**, then $V\psi$ with appropriately chosen parameters α is part of a SU*(4) (transition) matrix element and may be equally well determined by orthogonality relations of the group SU*(4). The physically (and historically) interesting feature, however, emerges in the given coset decomposition and the integrated form of *V.* If we understand as above second quantization as a principle to only use symplectic irreps and transformations which ensures that we always work with a canonical system, the various actions of SU*(4) transformations in general will not respect the original irreducibility of the second quantized ψ with respect to the compact group USp(4). However, if we rewrite a group action. *G* of SU*(4) according to the polar decompositon

$$G = \exp(g)\,\exp(V), \qquad g \in \mathrm{usp}(4) \tag{10}$$

and use Eq. (9), the unit element $\sim$cosh α rescales the symplectic matrix element $\langle\cdot|\exp(g)|\cdot\rangle$, whereas the term $\sim V$ should vanish to maintain the (originally introduced) quantization structure of the system. Thus, with an appropriate choice of the parameters of the group element *G* with respect to physical observations, we may interpret the Dirac equation in the sense of Wigner (1962) in that the equation of motion only serves to suppress superfluous components of otherwise not irreducible representations. Here, the Dirac equation $V\psi = 0$ can be used phenomenologically to suppress unwanted SU*(4) components of a second quantized USp(4) irrep ψ which are introduced by transformations of the full Dirac algebra Γ in terms of the SU*(4) action *G*. Appropriately, use of the Dirac equation is a subordinate conceptual approach with respect to SU*(4) and its USp(4) subgroup representations, whereas the use of Lie group or Clifford theory allow us to treat the underlying projective geometry algebraically.

For now, however, it is very important that the Lie algebra isomorphism given above yields a *consistent* and well-defined mathematical framework for physical investigations *on all three* representation spaces, either by algebra isomorphisms or by contractions/projections. In the following, we focus on some consequences of a representation on complex spaces which allow us to understand hadronic structure in a very fundamental way.

4. HADRON REPRESENTATIONS

In the complex case, we represent relativistic Sl(2,**H**) transformations as above by a set of 4×4 complex matrices which is isomorphic to SU*(4). The idea of looking for cosets with respect to the maximal compact subgroup USp(4) automatically leads to the compact symmetry group SU(4), which is related to SU*(4) by Weyl's unitary trick. The correlated physics can easily be understood by identifying the parameters of the coset space with velocity components and discussing the low-energy (low-velocity) limit. In this limit, a difference between real and purely imaginary parameters in Eqs. (7) and (9) is related to the difference between transcendental and hyperbolic functions in the integrated form, so that the physics described by these two theories is very similar for small absolute values of the respective parameters. Thus, in the low-energy regime of the spectrum, the compact symmetry group SU(4) can be expected to serve as a suitable ‘effective’ description of relativistic dynamics in terms of complex vector/spinor spaces and action of the non-compact group SU*(4) $\cong$ Sl(2,**H**). Moreover, it is the coset decomposition SU*(4)/USp(4) and the further subgroup chain which suggests we introduce positive/additive masses as projection parameters into the formerly homogeneous theory based on SU*(4) and Sl(2,**H**), i.e., of exact relativistic interactions of elementary particles. This relation can explain why SU(4) is a good symmetry group for *low* energies in Wigner’s supermultiplet scheme (Wigner, 1937) although there the nucleon is identified within another representation. However, using SU(4), we are able to overcome most of the cited problems within phenomenological hadron models. We identify the nucleon *and* delta resonances within the third-rank symmetric spinorial representation $\Psi^{\alpha\beta\gamma}$ (Dahm and Kirchbach, 1995, Dahm, 1997a) because these 20 states saturate the Adler–Weisberger relation almost completely. Thus, isospin as well as chiral transformations can be identified as parts of SU(4) transformations acting on the higher spinorial irrep $\Psi^{\alpha\beta\gamma}$ and SU(4) can be used to classify the low-lying particles in the spectrum and describe their ‘parity partners’. There is no need to work with Goldstone or nonlinear realizations from the beginning, but we may use an SU(4) Wigner–Weyl realization with a well-defined Yukawa-like coupling scheme. The observed spin/isospin degrees of freedom of the 15 bosons π, ρ, and ω fit perfectly into the regular representation **15** of SU(4), and the rank 3 of SU(4) allows us very naturally to define three group invariants of order 2, 3, and 4 by means of Casimir operators. The orders 2 and 4 are similar to the behavior of a phenomenological Mexican hat potential introduced in spontaneously broken models, whereas terms of order 3 are already known from supersymmetric models. SU(4), when decomposed according to its SU(2) $\times$ SU(2) subgroup, yields small isospin breaking (Dahm, 1996) which has been discussed in literature, and it determines the N–Δ transitions very well by using only *one* coupling parameter which at the end emerges to be equal to unity (Dahm, 1996).

Conversely, the occurrence of SU(2) or even of ‘chiral’ SU(2) $\times$ SU(2) structures in physical observations is no surprise if we use symmetries described by higher dimensional algebras/groups. Such properties can be derived directly from Dynkin diagrams when investigating, e.g., SU(4) $\cong$ A_3. Each dot of the Dynkin diagram corresponds to a separate triplet of SU(2) generators, and triplet operators belonging to dots which are not connected by a direct line do commute (Joseph, 1970). Thus, the simplest compact Lie groups which naturally occurs as subgroups when dealing with higher dimensional algebras will *necessarily* be SU(2) or ‘chiral’ SU(2) $\times$ SU(2) as soon as the rank l of the algebra is $l \geq 3$. The theoretical problem to realize an SU(2) or SU(2) $\times$ SU(2) symmetry structure in order to describe the particle spectrum is isomorphic to the technical problem to realize an appropriate coset reduction of the dynamical (noncompact) symmetry group with respect to 'one or two dots' of the Dynkin diagram. We thus obtain all the features introduced by hand into chiral models, and we may use well-defined coset decompositions and subgroup chains to split the supermultiplet structures. However, there is no further need for artificial requirements like massless pions and additional explicit symmetry breaking in terms of PCAC to fit the theory to physics.

The most crucial feature of our approach, however, is the relation of SU(4) spinorial structure to the quark structure of hadrons. Note once more that using SO*(4) and acting on complex representation spaces only serves to represent quaternionic projective transformations of Sl(2,**H**). Appropriately, we use the symmetric rep $\Psi^{\alpha\beta\gamma} \cong \mathbf{20}$ to describe the nucleon/delta multiplet and the vectorial rep $M^{\alpha\beta} \cong \mathbf{15}$ to describe bosons. Thus, the complex representation of Sl(2,**H**) enforces a 'quark structure' of hadrons naturally in that the nucleon/delta comprises three (symmetrized) fundamental spinor fields and the boson (vector) representation consists of a fundamental spinor and a complex conjugate one due to the vector character of **15** in SU(4) and SU*(4). *It is only the very choice of complex representation space which causes the hypothesis of separately existing quark structures in order to explain nothing the (complex) spinorial structure of the (quaternionic) hadron representations*. Starting with complex representation spaces, a relativistic theory built from scratch on $\mathbf{C}^4$ and willing to describe hadronic particle interactions *necessarily* has to end up with the known spinorial 'quark' structure in order to correctly represent a quaternionic projective geometry on complex representation spaces. It is obvious that our description has no need for a physical interpretation of QCD gauge groups and appropriate vector bosons to glue fundamental spinor representations together. Such approaches are subordinate concepts which try to cover only few facets of the underlying full geometrical concept by certain, sometimes arbitrary physical identifications. For example, in the case of a vector representation in $\mathbf{R}^3$ when investigating SO(3) transformations, nobody would try to establish a additional spinorial substructure, although one may equally well choose an equivalent complex representation with two appropriate spinorial indices by switching to complex representation spaces and by using SU(2) transformations. However, one has to be very careful with the ranges of the respectively underlying geometrical interpretation and the meaning of 'physical' parameters and observations. With respect to our quaternonic approach, we suggest to benefit from the existence of appropriate representation spaces as purely *mathematical*, but adequate (equivalent) tools. In addition, it is important to be very careful when trying to establish identifications with respect to physical processes and observations.

In this context, we observe one more striking feature when dealing with the SU*(4) low-energy representation. As mentioned in Section 1, it is possible to represent nucleons by fundamental Dirac spinors ψ if we 'add some correction terms' like the Pauli term to the transformation of the fundamental representation. This, of course, leads immediately to the question why algebraically the fundamental rep $\Psi \cong \mathbf{4}$ is a suitable (effective) approximation to the third-rank spinorial representation $\Psi^{\alpha\beta\gamma} \cong \mathbf{20}$. If we investigate the root diagrams of the fundamental rep **4** (see Fig. 1, left) and the third-rank spinorial rep **20** (Fig. 1, right), the tetrahedrons **4** and **20** have globally the *same* geometrical structure and thus (up to a scale factor) geometrically the same overall behavior. The differences between the two representations only emerge when we try to look closer into their respective structures, e.g., by introducing a common scale measure via (electromagnetic) interactions which couple the representations **4** and **20**. Depending on the energy, the tetrahedron **20** shows its threefold symmetric spinorial substructure and thus shows a behavior different from **4**, but in analogy to physical observations respectively their usual interpretation in terms of additional 'quark substructures'.

A further benefit of using the complex representation SU*(4) of quaternionic projective theory is its much more general approach to understand 'chiral symmetry' and its covering of known models of SU(2) $\times$ SU(2) Chiral Dynamics. As discussed above, it is already apparent from the Dynkin diagram that A_3 has a commuting SU(2) $\times$ SU(2) subgroup. To obtain directly the known nonlinear chiral realizations, we represent a homogeneous twofold

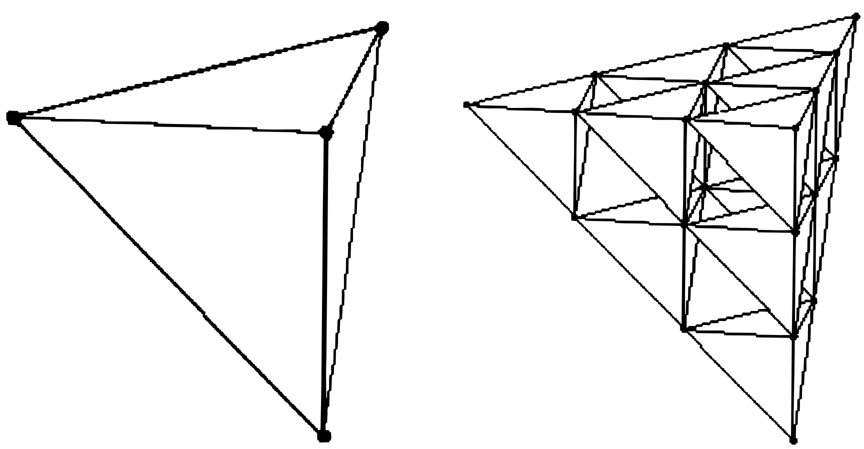

Fig. 1. Fundamental rep **4** and thlrd-rank symmetric spinorial rep **20** of SU(4).

quaternionic transformation as a fractional transformation (for detailed discussion see (Dahm, 1995)),

$$
\begin{aligned}
f(q) &= \frac{aq+b}{cq+d} \\
&:= \left. aq+b \right| \\
&\quad \left| cq+d \right. \\
&\equiv (aq+b)(cq+d)^{-1}, \qquad a, b, c, d, q \in \mathbf{H} \qquad (11)
\end{aligned}
$$

Restricting the transformation $f(q)$ by $b = c = 0$ and $q = U$, $a, d \in$ SU(2) $\cong$ U(1,**H**) to unit quaternions, we can introduce the ('nonlinear') parametrizations

$$U = \exp(\vec{q} \cdot \vec{\varphi}), \qquad a = \exp(\vec{\epsilon}_R \cdot \vec{q}), \qquad d = \exp(\vec{\epsilon}_L \cdot \vec{q}) \qquad 12$$

of these unit quaternions as well as of 'left' and 'right' transformations determined by $\vec{\epsilon}_L$ and $\vec{\epsilon}_R$. Conservation of the norm $U^{+}U$ and a reinterpretation of the power series of the parameters $\vec{\varphi}$ in terms of linear SO(4) group representations leads exactly to the linear σ-model as the already integrated form of nonlinear chiral models. Weinberg's nonlinear pion transformation laws (Weinberg, 1968) are obtained directly from Eq. (11) as infinitesimal transformations. This can be seen by simply expanding $f(q)$ in terms of the six transformation parameters $\vec{\epsilon}_R$ and $\vec{\epsilon}_L$ given in Eq. (12). Appropriately, it is apparent that the 'chiral' structure of these models only stems from the noncommutativity of quaternions in the quaternionic projective transformation (11) and that there is no meaningful generalization to SU(n) $\times$ SU(n) for arbitrary n without severe changes of the underlying geometrical principle. A generalization of our approach can be achieved on the basis of the four division algebras with unit element in order to avoid zero divisors, e.g., by identifying the quaternions within split octonions or by direct investigations of an appropriate octonionic geometry with all its difficulties.

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